NOTES AND QUERIES.

PROFESSOR DE MORGAN favours us with the following problem, and, as will be observed, invites solutions of it:—

For abbreviation, let $(1+r)^n$ be represented by $1+1_n r+2_n r^2+3_n r^3+\ldots$

The fractions whose numerators are $(1+r)^n-1$, $(1+r)^n-1-1_n r$, $(1+r)^n-1-1_n r-2_n r^2$, $(1+r)^n-1-1_n r-2_n r^2-3_n r^3$, &c., and whose denominators are r, r^2, r^3, r^4, &c., represent the accumulations of the annuities 1, 1, 1, &c. for n years; 1, 2, 3, &c. for $n-1$ years; 1, 3, 6, &c. for $n-2$ years; 1, 4, 10, &c. for $n-3$ years; &c.

Required a proof; *the calculus of differences and all summation of series involving powers being excluded.*

CORRESPONDENCE.

THE INTEREST QUESTION.

To the Editor of the Assurance Magazine.

SIR,—In your last Number* of the *Assurance Magazine,* "A Young Associate," who seems to be inspired with the doctrines of *a well known writer,* has attempted to defend the theory of the latter—first, by pointing out to me the solution to the problem specified at the conclusion of my letter in No. XII. of your *Magazine;* and, secondly, by directing my attention to the work of Mr. Rouse, who, like myself, in the opinion of the writer, erred in assuming the geometrical mean of $(1+d)^{\frac{1}{2}}$ between 1 and $(1+d)$, instead of the arithmetical mean of $\left(1+\frac{d}{2}\right)$, *as in common use.*

I venture to suppose that, like myself, all truly mathematical readers will consider that in cases of compound interest, which forms a strictly *geometrical* series, the upholding an arithmetical mean against a geometrical mean is in itself *mean* indeed, and *means* nothing; as will be demonstrated presently. Moreover, to cite *common use* in a question of a purely mathematical character, is really absurd, and no *common* sense!

Mr. Farren, in No. XI. of your periodical, inserted a paper wholly directed against Simpson and Dodson, for having imagined that De Moivre assigned $1-i$A as the present value of £1 payable at the end of the year of death, and for having accordingly adopted that value. Mr. Farren states, that "there are reasons (?) for imagining that, in *one case at least,*† this celebrated analyst fell into no such error, but correctly (?) assigned $1-i$A as the present value of £1 due at the *beginning* of the year of death,

* This communication was intended for the January Number, but was received too late for insertion in it.—ED. *A. M.*

† I suppose that by the indefinite expression "*one case at least,*" Mr. Farren meant *this case,* the one at issue; but preferred saying *one,* as an indication that on all other subjects De Moivre *did* fall into error—an inference which, however, must be made with caution.

and bearing interest until actual payment ensued." Mr. Farren then proceeds to prove the truth of his discovery, by quoting a certain paragraph from De Moivre, and adding thereto a commentary of his own; whereby he fancies to have established an indisputable fact, that half a year's interest above any number of years is to be obtained by multiplying the amount accumulated at the beginning of that half year by $\frac{1}{2}$.

I, on my part, in No. XII. of your *Magazine*, have proved from the very work of De Moivre that Mr. Farren's supposition was unfounded; and that, on the contrary, De Moivre directed to value a portion of a year by adding the logarithm of the amount as accumulated at the beginning of the broken year to a similar portion of the logarithm of one year's interest: and, in order to substantiate my refutation, I concluded my letter with the problem on compound interest, whereby I had every reason to anticipate that your readers might discover that not only did Mr. Farren fail in his attempt to give a wrong interpretation to De Moivre, but that actually there is *no* other mode for finding the true mathematical value. However, it appears that your correspondent, the admirer of a *well known writer*, failed to comprehend my intention with the problem in question, and accordingly informs me that all English elementary treatises on algebra demonstrate that if $s(1+d)^x$ is to equal a, that x, as a period of duration, will equal the logarithm of a divided by the logarithm of $s(1+d)$. Well, the "Young Associate," who seems to be prejudiced against foreigners (no doubt a *common use* with him), may learn from his very solution, as derived from English elementary treatises, that both he and the well known writer *are* in error, and ought not to jump to conclusions too rapidly, without mature consideration. He must remember that from time immemorial it was a *common use* to teach that the sun revolved round our earth; and that not until the sixteenth century, Copernicus, a *foreigner*—I am proud to say, a Pole, too—turned the tables, in establishing the fact that the earth revolves round the sun, as also corroborated in the English elementary treatises.

Now to the subject:—

Problem I.—A sum of £1,000 is put out at compound interest at 3 per cent. per annum, for a period of $75\frac{1}{2}$ years. It is required to find the amount.

Solution.—According to Mr. Farren, and *common use*, the amount in 75 years is £9,178·92566, to which must be added half a year's interest, £137·68388=£9,178·92566 × 0·015; the sum of £9,316·60954 is the amount required.

Problem II.—A sum of £1,000 is put out at compound interest at 3 per cent. per annum, until the amount reaches the sum of £9,316·60954. To find the period of duration.

Solution.—According to all elementary treatises, English or foreign, as also pointed out by "A Young Associate," the logarithm of £9,316·60954 is to be divided by the logarithm of £1,000 (1·03). Thus

$$\frac{\cdot 9692578945}{\cdot 0128372247} = 75\cdot 50369 \ldots \text{ the period required.}$$

Everyone will perceive that in the two foregoing problems the very identical values are employed; and yet in the former the period x is fixed

to be neither more nor less than 75·5 years, whilst, according to *common use*, the same term in the second problem amounts to 75·50369 years. Thus, according to Mr. Farren, "A Young Associate," and others, 75·5 = 75·50369!

I need scarcely observe, that the difference between the two values will always increase as may increase the period x. It is manifest that, if the portion of the last year be fixed as a finite, the quotient obtained by dividing the logarithm of the amount by the logarithm of i must invariably consist of a finite quantity, which can never be the case if the interest of the broken period is allowed to be obtained by multiplying the amount at the beginning of the year by $\frac{i}{2}$.

I enclose herewith a letter for publication, addressed to me by a celebrated mathematician, late actuary of the "Alliance," on the subject in dispute, whose opinion tends to confirm my own views.

I remain, Sir,
Your most obedient Servant,
HERSCHEL FILIPOWSKI.

Standard Life Office,
Edinburgh, Dec. 19, 1853.

"To H. Filipowski, Esq.

"My dear Sir,—Having in my reply to your letter of the 7th of last month expressed my reluctance to enter on the discussion of the question between yourself and other scientific gentlemen, and you having in reply to me still expressed a wish that I should give *you* my opinion of your ideas —I will, with a view of complying in part with your wish (but still with a desire to avoid being compelled to enter into a scientific discussion with gentlemen who may be disposed to maintain their own views, and to support those views by arguments, which may appear to themselves to be uncontrovertable), take up the subject, of small pecuniary importance, but of interesting scientific accuracy; and hope to avoid any personalities, by considering the question of compound interest as a question of science, agitated from the time commencing with the excellent D'Alembert, and ending with Milne, whose death is but a recent loss to the scientific world, and whose name alone is sufficient to give a bias to opinions, if they cannot be refuted by argument.

"I consider the colloquial term of 'compound interest' to be vaguely interpreted. I consider the commercial expressions of a capital placed out, for instance at 4 per cent. per annum, payable in half-yearly interest, when it refers to a capital to be placed at compound interest, by no means to be accurately interpreted by a capital placed at compound interest at 4 per cent. per annum; but to be properly expressed by a capital placed at the semi-annual interest of 2 per cent., or by a capital placed at the annual rate of 4·04 per cent. compound interest. And with this definition: 2 per cent. half-yearly interest would be perfectly consistent; and then, if n be a whole number, and the annual rate of interest be (as it would be, in fact) 4·04 per cent. per annum, the amount in n years, with compound interest, of £100, would be $\overline{1{\cdot}0404}^{n} \times$ £100; and if n were the fraction $\frac{1}{2}$, the expression would still apply, and become $\overline{1{\cdot}0404}^{\frac{1}{2}} \times$ £100, as this would evidently be £102, agreeing with the proposed fact. So that, if

£100 be put to compound interest at the annual rate of 4·04 per cent., and it were recalled at the expiration of half a year, the amount with interest would be £102, and would be less actually than if the half year's interest were calculated on the scale of simple interest, estimating the annual interest at 4·04 per cent., which is real annual interest per annum, proposed (a fact, I think, considered by Mr. Milne as remarkably inconsistent) on the supposition at least that money is at all times and almost immediately convertible into capital—which it generally is, though there may be occasional short periods at times when money cannot be so turned; and such periods would not vitiate the universality of the definition, but alter the condition of the case to be stated, which would allow of $\overline{1+r}^n$ expressing the amount of unity placed at compound interest for n years, at the annual rate of r per unit, whether n were whole, broken, or negative; and I think even those persons who would aim at a distinction between the cases of n integer and n fractional would allow, that if n were negative $= -m$, m being an integer, that $\overline{1+r}^n = \overline{1+r}^{-m}$ would express the value of the sum which had been put to compound interest; for m years, at the rate of r per annum, per unit, would amount to unity: or, in other words, that the present value of unity to be received in m years is truly expressed by $\dfrac{1}{\overline{1+r}^m}$. And I observe, that the semi-annual interest of 2 per cent. on a given sum would give 2 per cent. for the first half year; and that, that interest being converted into capital, the whole capital would give 2·04 per cent. for the second half year upon the original capital, though it would be but 2 per cent. on the capital increased by the first half year's interest; and that the total annual interest on the original capital would be 4·04 per cent. I therefore agree *entirely* with D'Alembert, and disagree entirely with Milne's view, with regard both to his objection to D'Alembert's views, and with his remark on Smart's tables. There may be cases in which the law might take different views to those of D'Alembert; but should there be such cases, they would not interfere with the question under my consideration, but would change the conditions of the question.

"I will conclude my letter with a case which might occur, to show that D'Alembert's opinion is right. There are three persons—A, B, C; A is a lender, B is an original borrower, and C an eventual borrower. B borrows of A £1000 for one year certain, at 5 per cent. interest per annum, and A would have required a higher interest, had it not been for the interference of the usury laws. At the expiration of a half year, B is desirous to pay back the money to A, together with the interest which shall be due upon it; but A objects, and states that the loan was intended to be for a year; and he does not know if, in case he should allow B to return capital with interest, he should be able to lend the money again, at the same rate, and to a person whom he considered as responsible and as trustworthy as B; but to oblige B, if he would engage to find a third person, C, for whom B would stand security, and C would take the money of A which is due to him from B at the same rate of interest, he would agree to comply with B's wishes: in consequence of which, B pays A what is due to him, and A immediately lends it to C. At the expiration of the half year from that period—that is, at the expiration of one year from the original period—A calls on C for the sum due, but unfortunately C cannot pay him; A therefore claims of B the capital and interest which B guaranteed C to pay him, and brings an

action for that claim. The question is, does he not require usurious interest of B?

"Now, calculation on Mr. Milne's hypothesis gives that B in the subsequent agreement pays A, at the expiration of the first half year, £102. 10*s*., and that A then lends to C £102. 10*s*., at £2. 10*s*. per cent. interest for the half year, which is required to complete the whole year: for were he only to lend him £100, A would leave £2. 10*s*. lying idle; and consequently, at the expiration of the year from the original term, C will have to pay A £105. 1*s*. 3*d*., being 1*s*. 3*d*. beyond the legal interest for which the action is brought against B. But on D'Alembert's hypothesis, B will have had only to pay at the expiration of the first half year $£100 \times \overline{1 \cdot 05}^{\frac{1}{2}}$; and C afterwards, for the continuance of the loan to him, and for which B is bound, $\overline{100} \times \overline{1 \cdot 05}^{\frac{1}{2}} \times \overline{1 \cdot 5}^{\frac{1}{2}} = £105$.

"Should these remarks tend to lessen the difficulty attaching to the case, I should be glad; but if I should be required to reply to objections which may be made to them, I feel that I should be obliged to decline doing so, and will hope that I should not be thought discourteous on that account.

"Yours truly,

"Benj. Gompertz.

"152, King's Road, Brighton,
"3 Nov., 1853."

CALCULATION OF THE ODDS OF THROWING ANY SPECIFIED NUMBER WITH TWO, THREE, FOUR, OR MORE DICE.

To the Editor of the Assurance Magazine.

Sir,—Some persons have supposed that the doctrine of probability rather fosters than discourages habits of gambling. No doubt the error of such a supposition arises from the known facility with which its principles can be applied to games at cards and dice. It has, however, been employed to expose the nefarious practices of many, who have developed very alluring though dishonest and fatal schemes for realizing money; and through the authority and influence of your *Journal*, the science of probability might be turned to some account in exposing those pernicious practices that are of nightly occurrence in many establishments in London, especially at the West End. The uninitiated and unwary, who seek amusement in these dens of infamy, might at all events be put on their guard, by having in their possession the *true odds* in every case where betting is resorted to on games of chance; and at the same time the usefulness and importance of your *Magazine* would be considerably augmented. With this view I have made the following calculations; and at a future time, I may direct my attention to other forms and shapes under which this insidious and dangerous practice presents itself.

When the throwing is with two dice, that are homogeneous and dynamically accurate (which is *never* the case in gambling houses), the probability of throwing either of the numbers

3 or 11 is $\frac{2}{36}$;	the odds against are	17 to	1
4 or 10 is $\frac{3}{36}$	,, ,,	11 to	1